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Potential symmetry and invariant solutions of Fokker-Planck equation in cylindrical coordinates related to magnetic field diffusion in magnetohydrodynamics including the Hall current

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Abstract. Lie groups involving potential symmetries are applied in connection with the system of magnetohydrodynamic equations for incompressible matter with Ohm's law for finite resistivity and Hall current in cylindrical geometry. Some simplifications allow to obtain a Fokker-Planck type equation. Invariant solutions are obtained involving the effects of time-dependent flow and the Hall-current. Some interesting side results of this approach are new exact solutions that do not seem to have been reported in the literature.

PACS. 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 52.30.Cv Magnetohydrodynamics (including electron magnetohydrodynamics) – 02.30.Jr Partial differential equations – 52.65.Ff Fokker-Planck and Vlasov equation

1 Introduction

Recently, Khater et al. [1–3] have analyzed the generalized one-dimensional Fokker-Planck equation (FPE) and the inhomogeneous NL diffusion equation through the application of the potential symmetries.

For the general and normalized expressions of the equations of magnetohydrodynamics (MHD) we refer to reference [3] as well. However, we add here some general considerations on the utility of obtaining solutions of the MHD equations, especially with the solution method considered here. MHD deals with the study of electromagnetic effects on conducting fluids and was mainly created by Alfven around 1940 (Nobelprize in physics 1970). Due to some (small) simplifications one may reduce the electromagnetic set of equations just to one equation of evolution involving the magnetic field only (however, linked to the fluid equations through the fluid velocity). Ideal MHD neglects dissipative effects, but may be extended to include resistivity, viscosity, the Hall current, etc., enlarging the domain of applications tremendously, but increasing the difficulty to obtain solutions and the stability analyses of those solutions very much too. MHD is essential in plasma physics and astrophysics (e.g. generation of magnetic fields in planets, Sun and stars by dynamo action, i.e. kinetic energy converted to magnetic energy). MHD determines the shape and fate of conducting fluid configurations pervaded by magnetic fields. The importance of obtaining exact solutions is evident. The strategy is usually to determine first equilibria or steady states or even time dependent states. Next one may investigate the stability of these configurations. The use of potential symmetries and invariant solutions is particularly interesting as this opens the way to several solutions or even classes of solutions, which, moreover, may allow comparisons in e.g. stability analyses. A previous paper, Khater et al. [3], was dealing with Cartesian geometry. However, in the laboratory (e.g. fusion research) and in nature (e.g. magnetic flux tubes like filaments and protuberances on the Sun) the configurations have often approximately a cylindrical shape. Here we investigate three dimensional cylindrical cases, again leading to a FPE, however, more involved, but more useful. For a brief exposition of the potential symmetries and of the equations of magnetohydrodynamics MHD: see [3].

Some interesting side results of the present study are new exact solutions that do not seem to have been reported in the literature.

This paper is organized as follows: Section 2 is devoted to the basic MHD equation. Section 3 deals with the determination of the potential symmetries. In Section 4, we analyze the invariant solutions of the MHD equations for various cases corresponding to physically interesting situations. Section 5 gives the conclusions.

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2 Basic MHD equations in cylindrical coordinates

We consider the equations of incompressible MHD including the Hall effect [3]:

$$\frac{j}{\sigma} = E + \frac{1}{c}\,\omega \times B - \frac{m_j}{\rho q c} j \times B - \nabla\left(\frac{m_i}{\rho q} p_e\right) \qquad (2.1)$$

$$\frac{\partial \omega}{\partial t} + (\omega \cdot \nabla)\,\omega = -\frac{1}{\rho}\nabla p + \frac{1}{c} j \times B + \nu \nabla^2 \omega, \qquad (2.2)$$

where E, B, ω, j, p, p_e, ρ, ν, σ, m_i, q and t stands as usual for the electric field, magnetic field, plasma velocity, electric current,plasma pressure, electron pressure, mass density, kinematic viscosity, electrical conductivity, mass of plasma ions, charge of plasma ions and time respectively, for details: see reference [3].

3 Determination of the potential symmetries

Consider a partial differential equation (PDE), of order m written in a conserved form: ([4] and references therein)

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i\left[x, u, u_1, u_2, ..., u_{m-1}\right] = 0 \qquad (3.1)$$

with $n \geq 2$ independent variables $x = (x_1, x_2, ..., x_n)$ and a single dependent variable u. For simplicity, we consider a single PDE — the generalization to a system of PDEs in a conserved form is straight-forward. The indices of u indicate the order of the derivative. If a given PDE is not written in a conserved form, there are a number of ways of attempting to put it in a conserved form. These include a change of variables (dependent as well as independent), an application of Noether's theorem [5], direct construction of conservation laws from field equations [6], and some combinations of them. Since equation (3.1) is in conserved form, then there is an $(n-1)$ exterior differential form equation (3.1) can be written as $dF = 0$, it follows that, there is an $(n-2)$ form Γ: $F = d\Gamma$.

Using some simplifications [3] we may put the equation of the evolution of flow in the MHD system for cylindrical coordinates (r, θ, z), which is a generalized FPE, in the following conservative form:

$$u_t - \left(\frac{1}{r^2} u_\theta + \frac{1}{r}\lambda_t \theta u\right)_\theta = 0 \qquad (3.2)$$

with λ a function of r and t; By considering a potential v as an auxiliary unknown function, the following system can be associated with equation (3.2):

$$v_\theta = u, \quad v_t = \frac{1}{r^2} u_\theta + \frac{1}{r}\lambda_t \theta u. \qquad (3.3)$$

It is well known that the homogeneous linear system, which characterizes the generators, is obtained from [7]

$$Y^{(1)}(v_\theta - u)\big|_s = 0, \quad Y^{(1)}\left(v_t - \frac{1}{r^2}u_\theta - \frac{1}{r}\lambda_t \theta u\right)\Big|_s = 0 \qquad (3.4)$$

which must hold identically.

Here, $Y^{(1)}$ is the operator:

$$Y^{(1)} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial \theta} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v} + \eta_1^{(1)}\frac{\partial}{\partial u_t} + \eta_2^{(1)}\frac{\partial}{\partial u_\theta} + \phi_1^{(1)}\frac{\partial}{\partial v_t} + \phi_2^{(1)}\frac{\partial}{\partial v_\theta}; \qquad (3.5)$$

$$\eta_1^{(1)} = \eta_t + (\eta_u - \tau_t)u_t - \tau_u u_t^2 - \xi_t u_\theta - \xi_u u_t u_\theta + \eta_v v_t - \tau_v u_t v_t - \xi_v u_\theta v_t,$$

$$\eta_2^{(1)} = \eta_\theta + (\eta_u - \xi_\theta)u_\theta - \tau_\theta u_t - \tau_u u_\theta u_t - \xi_u u_\theta^2 + \eta_v v_\theta - \tau_v u_t v_\theta - \xi_v u_\theta v_\theta,$$

$$\phi_1^{(1)} = \phi_t + (\phi_v - \tau_t)v_t - \tau_v v_t^2 + \phi_u u_t - \tau_u u_t v_t - \xi_t v_\theta - \xi_u u_t v_\theta - \xi_v v_t v_\theta,$$

$$\phi_2^{(1)} = \phi_\theta + (\phi_v - \xi_\theta)v_\theta - \xi_v v_\theta^2 + \phi_u u_\theta - \tau_\theta v_t - \tau_u u_\theta v_t - \tau_v v_\theta v_t - \xi_u u_\theta v_\theta.$$

where ξ, τ, η and ϕ are generators of the point symmetry group of corresponding auxillary system of the FPE.

The first equation of (3.4) becomes

$$\phi_2^{(1)} - \eta = 0 \qquad (3.6)$$

and from equation (3.5) we can rewrite equation (3.6) as

$$\phi_\theta + (\phi_v - \xi_\theta)v_\theta - \xi_v v_\theta^2 + \phi_u u_\theta - \tau_\theta v_t - \tau_u u_\theta v_t - \tau_v v_\theta v_t - \xi_u u_\theta v_\theta - \eta = 0, \qquad (3.7)$$

The second equation of (3.4) becomes

$$\phi_1^{(1)} - \frac{1}{r^2}\eta_2^{(1)} - \frac{1}{r}\theta\lambda_{tt}u\tau - \frac{1}{r}\lambda_t \xi u - \frac{1}{r}\lambda_t \theta \eta = 0 \qquad (3.8)$$

and from equation (3.5) we can rewrite equation (3.8) as

$$\phi_t + (\phi_v - \tau_t)v_t - \tau_v v_t^2 + \phi_u u_t - \tau_u u_t v_t - \xi_t v_\theta - \xi_u u_t v_\theta - \xi_v v_t v_\theta - \frac{1}{r}\lambda_t \theta\eta - \frac{1}{r}\lambda_t u\xi - \frac{1}{r^2}\left[\eta_\theta + (\eta_u - \xi_\theta)\,u_\theta - \tau_\theta u_t - \tau_u u_\theta u_t - \xi_u u_\theta^2 + \eta_v v_\theta - \tau_v u_t v_\theta - \xi_v u_\theta v_\theta\right] - \frac{1}{r}\theta\lambda_{tt}\tau u = 0. \qquad (3.9)$$

On substituting v_θ by u, and v_t by $\frac{1}{r^2}u_\theta + \frac{1}{r}\lambda_t \theta u$ in equations (3.7) and (3.9), we get:

$$\tau = \tau(t), \quad \xi = \xi(\theta, t), \quad \phi_u = 0 \qquad (3.10)$$

$$\phi_\theta - \eta + (\phi_v - \xi_\theta)u = 0 \qquad (3.11)$$

$$\phi_v - \eta_u - \tau_t + \xi_\theta = 0 \qquad (3.12)$$

$$\phi_t - \frac{1}{r^2}\eta_\theta - \frac{1}{r}\lambda_t \theta\eta + \left[\frac{1}{r}\lambda_t \theta\,(\phi_v - \tau_t) - \frac{1}{r}\lambda_t \xi - \xi_t - \frac{1}{r^2}\eta_v + \frac{1}{r}\lambda_{tt}\theta\tau\right] u = 0; \qquad (3.13)$$

with

$$\eta = f(\theta,t)u + g(\theta,t)v, \quad \phi = k(\theta,t)v, \tag{3.14}$$

where f, g and k are arbitrary smooth functions of θ and t. On solving the above system of equations (3.10)−(3.14), we get:

$$\tau = \tau(t), \quad \xi = \xi(\theta,t), \tag{3.15}$$

$$k_\theta - g = 0, \tag{3.16}$$

$$K - f - \xi_\theta = 0, \tag{3.17}$$

$$2\xi_\theta - \tau_t = 0 \tag{3.18}$$

$$k_t - \frac{1}{r^2}g_\theta - \frac{1}{r}\lambda_t\theta g = 0, \tag{3.19}$$

$$\frac{1}{r}\lambda_t\theta\tau + \frac{1}{r^2}g + \frac{1}{r^2}f_\theta + \xi_t + \frac{1}{r}\theta\lambda_t\xi_\theta + \frac{1}{r}\lambda_t\xi = 0, \tag{3.20}$$

$$g_t = \left(\frac{1}{r^2}g_\theta + \frac{1}{r}\lambda t\theta g\right)_\theta. \tag{3.21}$$

In solving the above system of equations (3.15)−(3.21), we confine our attention to physically interesting situations.

4 Invariant solutions

From now on, we will denote by $c_0 - c_{13}$ real arbitrary constants.
Let $\lambda = \frac{1}{2r}$
In this case, the infinitesimal symmetries are given by:

$$\left.\begin{aligned}
&\tau = -2r^2c_4e^{-\frac{t}{r^2}} + c_5,\\
&\xi = c_4e^{-\frac{t}{r^2}}\theta + c_3e^{-\frac{t}{2r^2}} - 2\sqrt{2r}c_0e^{\frac{t}{2r^2}},\\
&\eta = \left(\sqrt{2r}c_0e^{\frac{t}{2r^2}}\theta + c_1 - c_2\right)u + \sqrt{2r}c_0e^{\frac{t}{2r^2}}v,\\
&\phi = \left(\sqrt{2r}c_0e^{\frac{t}{2r^2}}\theta + c_1\right)v
\end{aligned}\right\}. \tag{4.1}$$

Then, we obtain point symmetries with the following generators:

$Y_1 : \tau = 0,\ \xi = -2\sqrt{2r}e^{\frac{t}{2r^2}},\ \eta = \sqrt{2r}e^{\frac{t}{2r^2}}\theta u + \sqrt{2r}e^{\frac{t}{2r^2}}v,$
$\quad \phi = \sqrt{2r}e^{\frac{t}{2r^2}}\theta v,$
$Y_2 : \tau = \xi = 0, \quad \eta = u, \quad \phi = v,$
$Y_3 : \tau = \xi = \phi = 0, \quad \eta = -u,$
$Y_4 : \tau = \eta = \phi = 0, \quad \xi = e^{-\frac{t}{r^2}}$
$Y_5 : \tau = -2r^2e^{-\frac{t}{r^2}}, \quad \xi = e^{-\frac{t}{r^2}}\theta, \quad \eta = \phi = 0,$
$Y_6 : \tau = 1, \quad \eta = \phi = \xi = 0$

and ∞-dimensional symmetry, which is a consequence of the linearity [8]. It is clear that, Y_1 is only a potential symmetry for equation (3.2).
For the potential symmetry Y_1, the characteristic system related to the invariant surface conditions reads:

$$v = c_7e^{\frac{-\theta^2}{4}}, \qquad t = c_6 \tag{4.2}$$

$$u = \left(c_8 - \frac{c_7}{2}\theta\right)e^{\frac{-\theta^2}{4}}. \tag{4.3}$$

If we assume $t = c_6 = z$ as a parameter, $c_7 = h_2(z)$, and $c_8 = h_1(z)$ in equations (4.2) and (4.3), we obtain:

$$u = \left(h_1(z) - \frac{h_2(z)}{2}\theta\right)e^{\frac{-\theta^2}{4}}, \tag{4.4a}$$

$$v = h_2(z)e^{\frac{-\theta^2}{4}}; \quad z = t. \tag{4.4b}$$

Now, to find the solutions F_E^*, we introduce equations (4.4a) in (3.2) obtaining:

$$h_1' - \theta\left(\frac{h_2'}{2} + \frac{h_2}{4r^2}\right) = 0 \tag{4.5}$$

which must hold for any value of θ.
From equation (4.5), we have the system as:

$$\left.\begin{aligned}
h_1' &= 0,\\
\frac{h_2'}{2} + \frac{h_2}{4r^2} &= 0
\end{aligned}\right\} \tag{4.6}$$

which on solving, yields

$$\left.\begin{aligned}
&h_1(z) = c_9,\\
&h_2(z) = c_{10}e^{-\frac{t}{2r^2}}
\end{aligned}\right\}. \tag{4.7}$$

Then, the family F_E^* is therefore:

$$u = \left(c_9 - \frac{c_{10}}{2}\theta e^{-\frac{t}{2r^2}}\right)e^{-\frac{\theta^2}{4}}. \tag{4.8}$$

Also, equation (4.4a) is a family of solutions of the first-order equation:
To find the solutions F_E, we introduce equations (4.4) in (3.3) obtaining the system as:

$$\left.\begin{aligned}
h_1 &= 0,\\
h_2' + \frac{h_2}{2r^2} &= 0
\end{aligned}\right\} \tag{4.9}$$

which on solving, yields

$$\left.\begin{aligned}
&h_1(z) = 0,\\
&h_2(z) = c_{11}e^{-\frac{t}{2r^2}}
\end{aligned}\right\}. \tag{4.10}$$

Then, the family F_E is therefore:

$$u = -\frac{c_{11}}{2}\theta e^{-\frac{1}{4r^2}(\theta^2r^2+2t)} \quad \text{(see Fig. (2))}. \tag{4.11}$$

It is clear that, F_E is enclosed in F_E^*, which are new solutions as far as we know.

Particular case.

If, $f = \lambda_t\theta$, $g = v^z = 0$, and $u = \Theta_\theta$ in equations (3.12)−(3.15) we obtain that

$$f_1(t) = 0, \tag{4.12}$$

$$u = u(t) \tag{4.13}$$

$$u_t = \frac{1}{r}\lambda_t u, \tag{4.14}$$

$$\lambda_{tt} + \frac{2}{r}\lambda^2 - \frac{1}{r^2}\lambda = 0 \qquad v_m = 1. \tag{4.15}$$

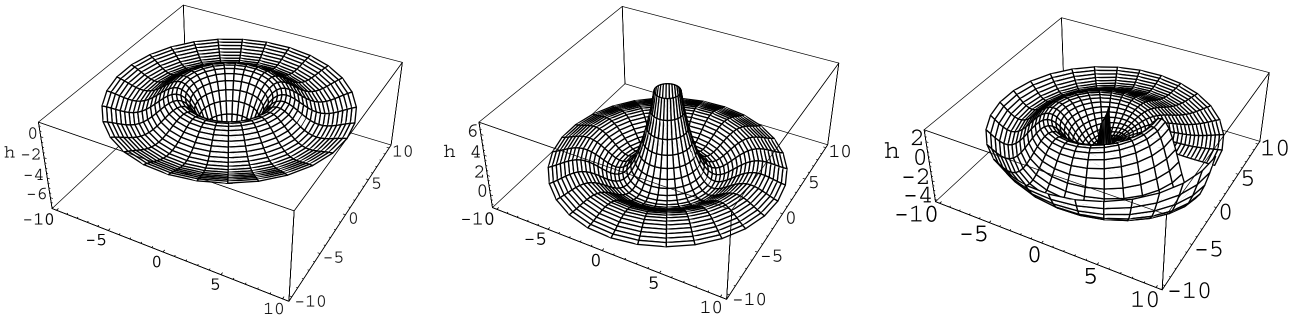

Fig. 1. (a) The magnetic field in the surface with $\mu_1 = 10, \mu_2 = 0.1$ and $m = 0.001$. (b) The magnetic field in the surface with $\mu_1 = -10, \mu_2 = 0.1$ and $m = 0.001$. (c) The magnetic field in the surface with $\mu_1 = 10, \mu_2 = 0$ and $m = 0.01$.

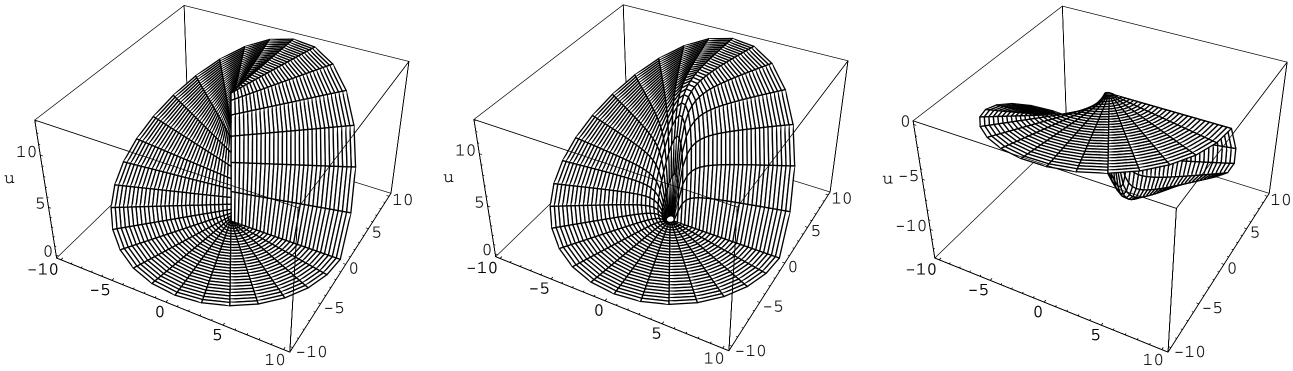

Fig. 2. (a) The solution for a Fokker-Planck in the surface with $c_{11} = -30$, $t = 0$. (b) The solution for a Fokker-Planck in the surface with $c_{11} = -30$, $t = 1$. (c) The solution for a Fokker-Planck in the surface with $c_{11} = 30$, $t = 1$.

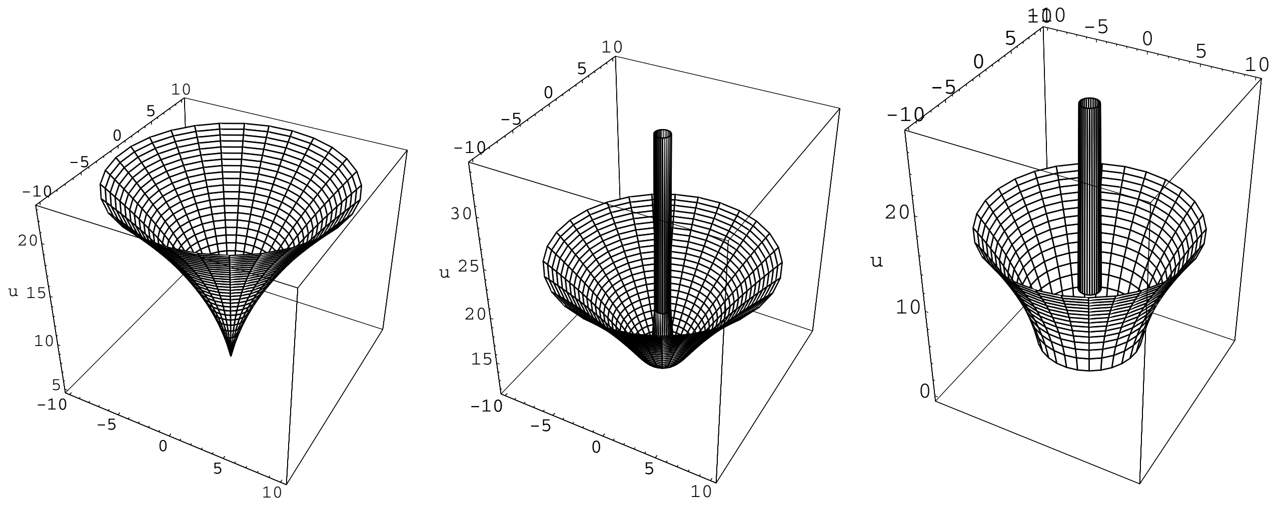

Fig. 3. (a) The solution for a Fokker-Planck in the surface with (particular case) $c_{13} = 5$, $t = 0$, $c_{12} = 1$. (b) The solution for a Fokker-Planck in the surface with (particular case) $c_{13} = 5$, $t = 1$, $c_{12} = 1$. (c) The solution for a Fokker-Planck in the surface with (particular case) $c_{13} = 5$, $t = 2$, $c_{12} = -10$.

Solving equation (4.15), yields

$$\lambda_t = \frac{1}{2r + c_{12} e^{-\frac{t}{r^2}}}. \tag{4.16}$$

Then, the family F_E^* is given by:

$$u = c_{13}\sqrt{2re^{\frac{t}{r^2}} + c_{12}} \quad \text{(see Fig. (3))}. \tag{4.17}$$

It is clear that, F_E is enclosed in F_E^*, which are new solutions as far as we know. In Figure 3, as in Figure 1, the solution becomes infinite on the axis of the cylinder. This may be avoided by considering a hollow conducting fluid cylinder. In some cases this may involve constraints due to boundary conditions, especially if the cylinder does not extend radially to infinity: then only sets of possible internal and external radii of the hollow cylinder will be allowed and not arbitrary ones.

5 Conclusion

In this paper, we made an analysis for the FPE with convection given by the plasma flow with finite electrical conductivity and Hall current. This method based on potential symmetries turns out to be an alternative, systematic and powerful technique for the determination of the solutions of linear or nonlinear PDEs, single or a system. The infinitesimals, similarity variables, dependent variables, and reduction to quadrature or exact solutions of the mentioned FPE (in cylindrical coordinates) for physically realizable forms of λ, u and the magnetic field induction h are also obtained.

The similarity solutions given here do not seem to have been reported in the literature. Some of these solutions are unbounded. However, one can deal with them as various methods have been elaborated to analyze the properties of unbounded (particularly explosive type) solutions of the Cauchy problem of quasilinear parabolic equations of type (3.2).

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